Digital Image Processing

Lecture 5
Image Enhancement in Spatial Domain

Fall 2010

Histogram Processing

- Histogram of a digital image with gray levels in the range [0,L-1] is a discrete function

$$h(r_k) = n_k$$

- Where
  - r_k : the k^{th} gray level
  - n_k : the number of pixels in the image having gray level r_k
  - $h(r_k)$: histogram of a digital image with gray levels r_k

Normalized Histogram

- dividing each of histogram at gray level r_k by the total number of pixels in the image, n

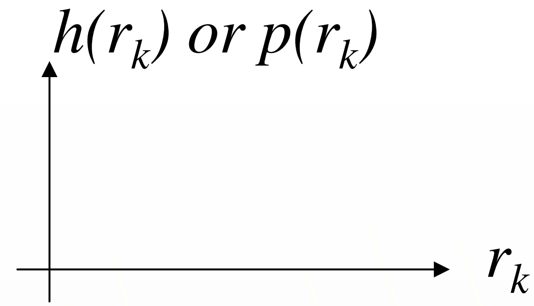
$$p(r_k) = n_k / n$$

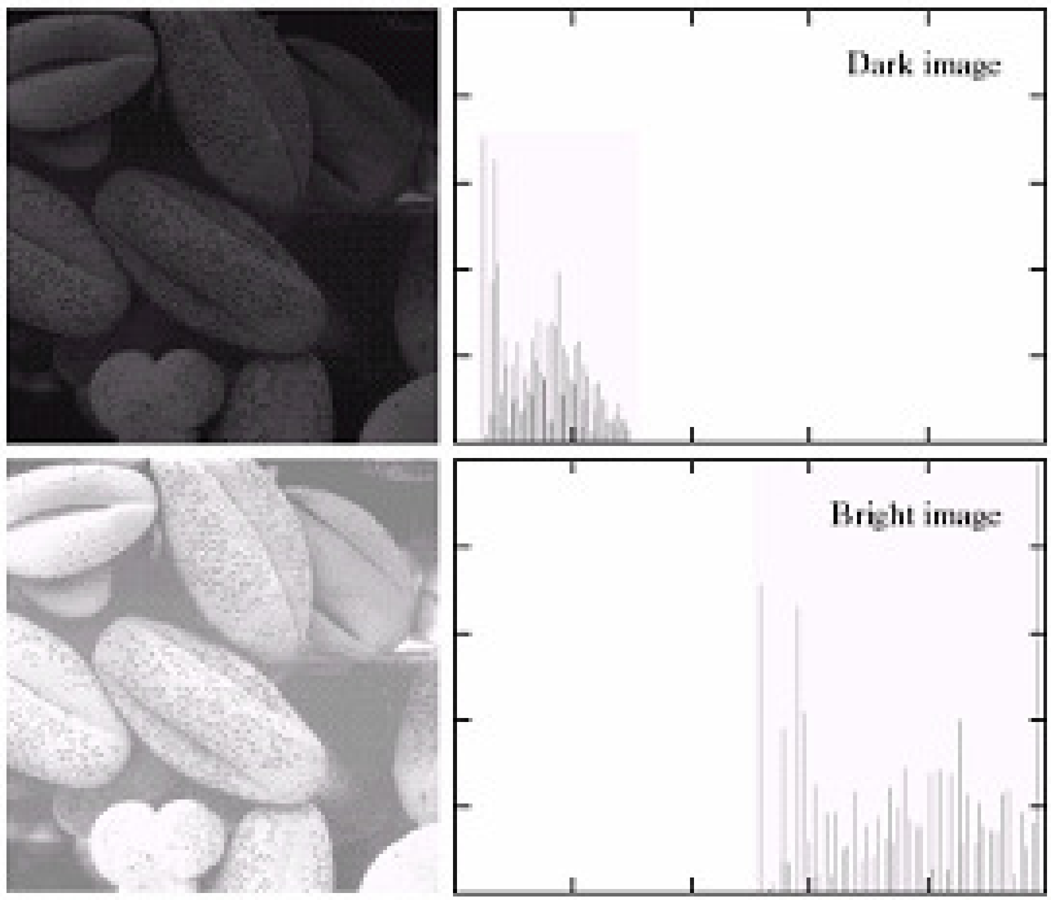
- For k = 0,1,…,L-1
- $p(r_k)$ gives an estimate of the probability of occurrence of gray level r_k
- The sum of all components of a normalized histogram is equal to 1

Histogram Processing

- Basic for numerous spatial domain processing techniques
- Used effectively for image enhancement
- Information inherent in histograms also is useful in image compression and segmentation

Example





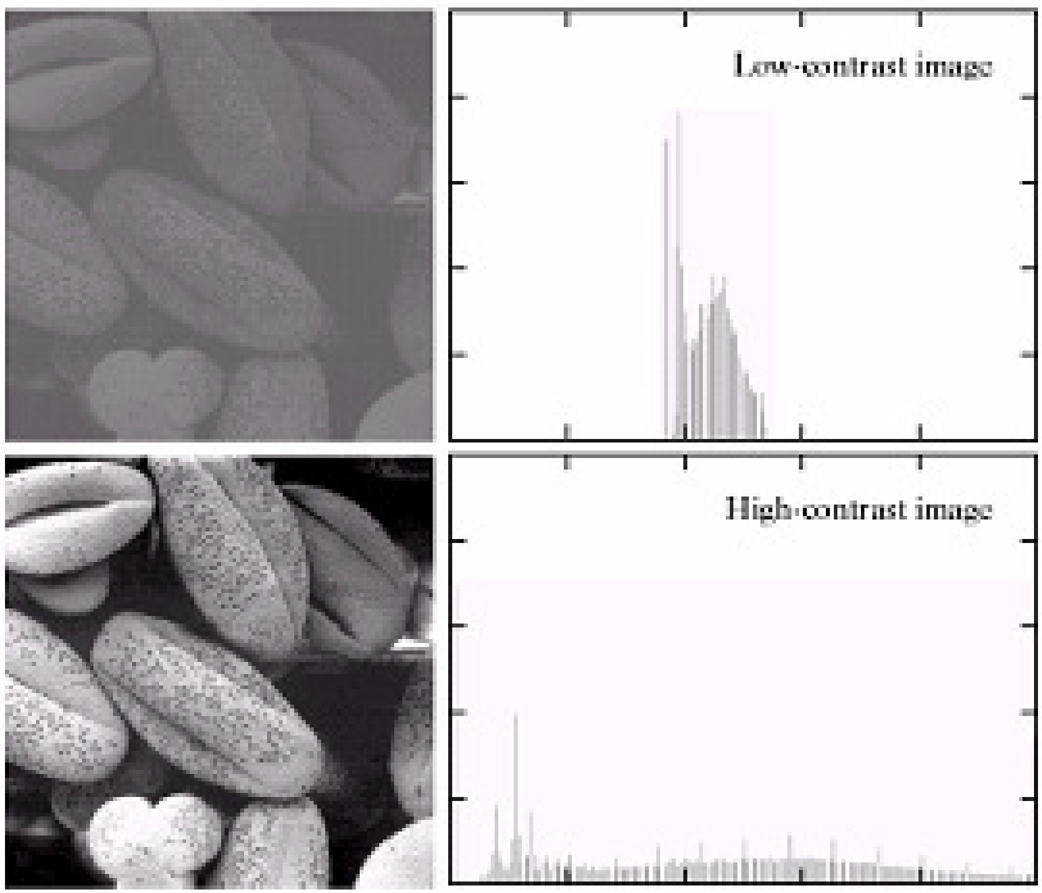
Dark image

Components of histogram are concentrated on the low side of the gray scale.

Bright image

Components of histogram are concentrated on the high side of the gray scale.

Example



Low-contrast image

histogram is narrow and centered toward the middle of the gray scale

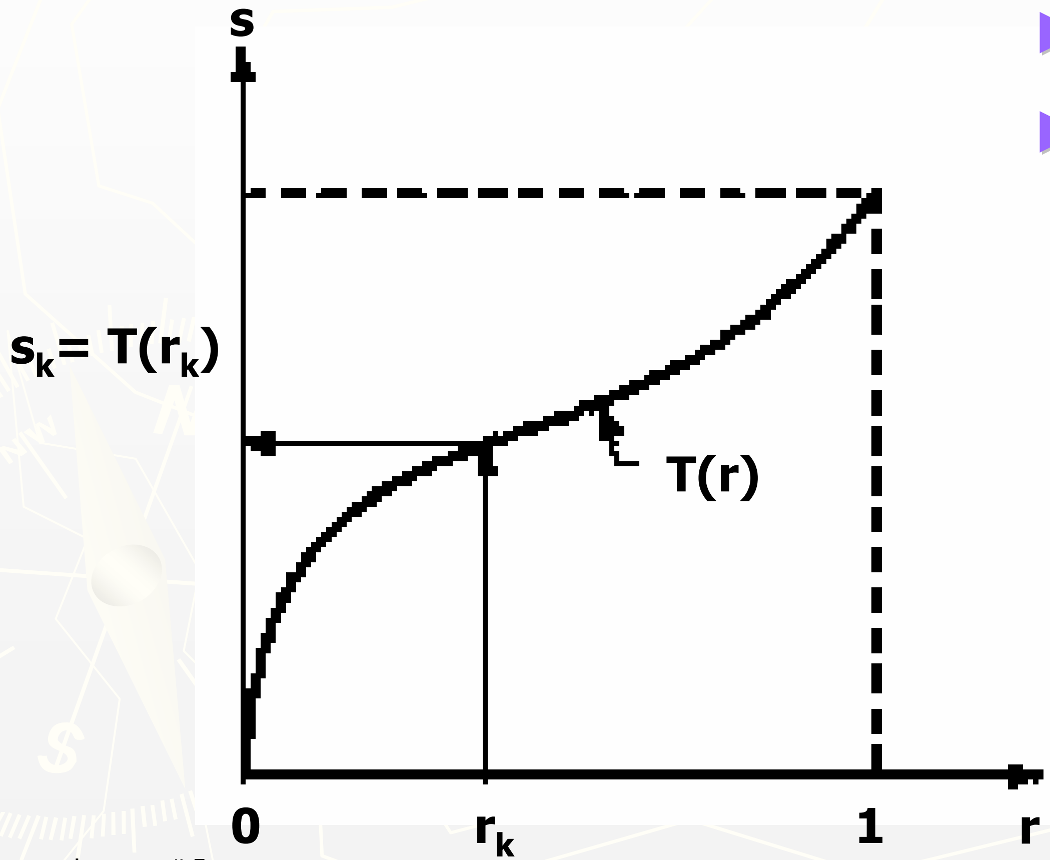
High-contrast image

histogram covers broad range of the gray scale and the distribution of pixels is not too far from uniform, with very few vertical lines being much higher than the others

Histogram Equalization

- As the low-contrast image's histogram is narrow and centered toward the middle of the gray scale, if we distribute the histogram to a wider range the quality of the image will be improved.
- We can do it by adjusting the probability density function of the original histogram of the image so that the probability spread equally

Histogram Equalization



$$s = T(r)$$

- Where $0 \leq r \leq 1$
- T(r) satisfies
  - (a). T(r) is single-valued and monotonically increasingly in the interval $0 \leq r \leq 1$
  - (b). $0 \leq T(r) \leq 1$ for $0 \leq r \leq 1$

2 Conditions of T(r)

- Single-valued (one-to-one relationship) guarantees that the inverse transformation will exist
- Monotonicity condition preserves the increasing order from black to white in the output image thus it won't cause a negative image
- $0 \leq T(r) \leq 1$ for $0 \leq r \leq 1$ guarantees that the output gray levels will be in the same range as the input levels.
- The inverse transformation from s back to r is

$$r = T^{-1}(s) \quad ; 0 \leq s \leq 1$$

Probability Density Function

- The gray levels in an image may be viewed as random variables in the interval [0,1]
- PDF is one of the fundamental descriptors of a random variable

Applied to Image

- Let
  - $p_r(r)$ denote the PDF of random variable r
  - $p_s(s)$ denote the PDF of random variable s
- If $p_r(r)$ and $T(r)$ are known and $T^{-1}(s)$ satisfies condition (a) then $p_s(s)$ can be obtained using a formula :

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|$$

Applied to Image

The PDF of the transformed variable s

is determined by

the gray-level PDF of the input image

and by

the chosen transformation function

Transformation function

- A transformation function is a cumulative distribution function (CDF) of random variable r :

$$s = T(r) = \int_0^r p_r(w)\,dw$$

where w is a dummy variable of integration

Note: $T(r)$ depends on $p_r(r)$

Cumulative Distribution function

- CDF is an integral of a probability function (always positive) is the area under the function
- Thus, CDF is always single valued and monotonically increasing
- Thus, CDF satisfies the condition (a)
- We can use CDF as a transformation function

Finding $p_s(s)$ from given $T(r)$

$$\frac{ds}{dr} = \frac{dT(r)}{dr}$$

$$= \frac{d}{dr}\left[\int_0^r p_r(w)\,dw\right]$$

$$= p_r(r)$$

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|$$

$$= p_r(r)\left|\frac{1}{p_r(r)}\right|$$

$$= 1 \quad \text{where} \;\; 0 \le s \le 1$$

Substitute and yield

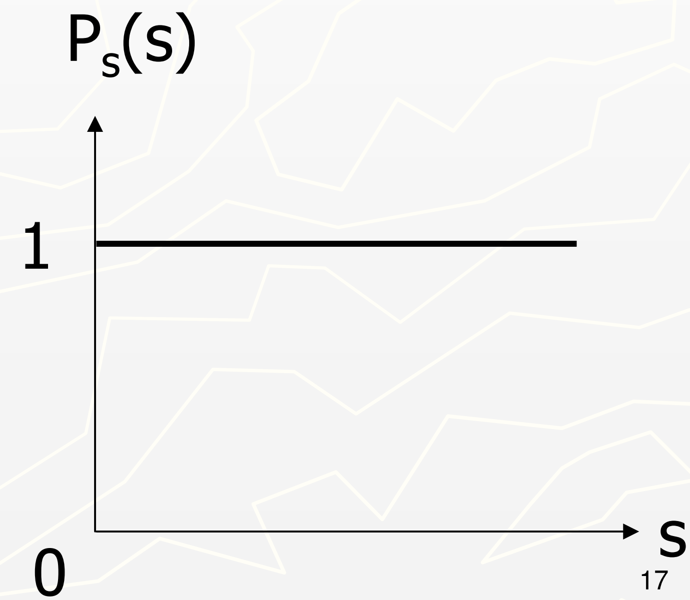
$p_s(s)$

- As $p_s(s)$ is a probability function, it must be zero outside the interval [0,1] in this case because its integral over all values of s must equal 1.
- Called $p_s(s)$ as **a uniform probability density function**
- $p_s(s)$ is always a uniform, independent of the form of $p_r(r)$

$$s = T(r) = \int_0^r p_r(w)\,dw$$

yields

a random variable s characterized by a uniform probability function



Discrete transformation function

- The probability of occurrence of gray level in an image is approximated by

$$p_r(r_k) = \frac{n_k}{n} \qquad where\ k = 0, 1, ..., L\text{-}1$$

- The discrete version of transformation

$$s_k = T(r_k) = \sum_{j=0}^{k} p_r(r_j)$$

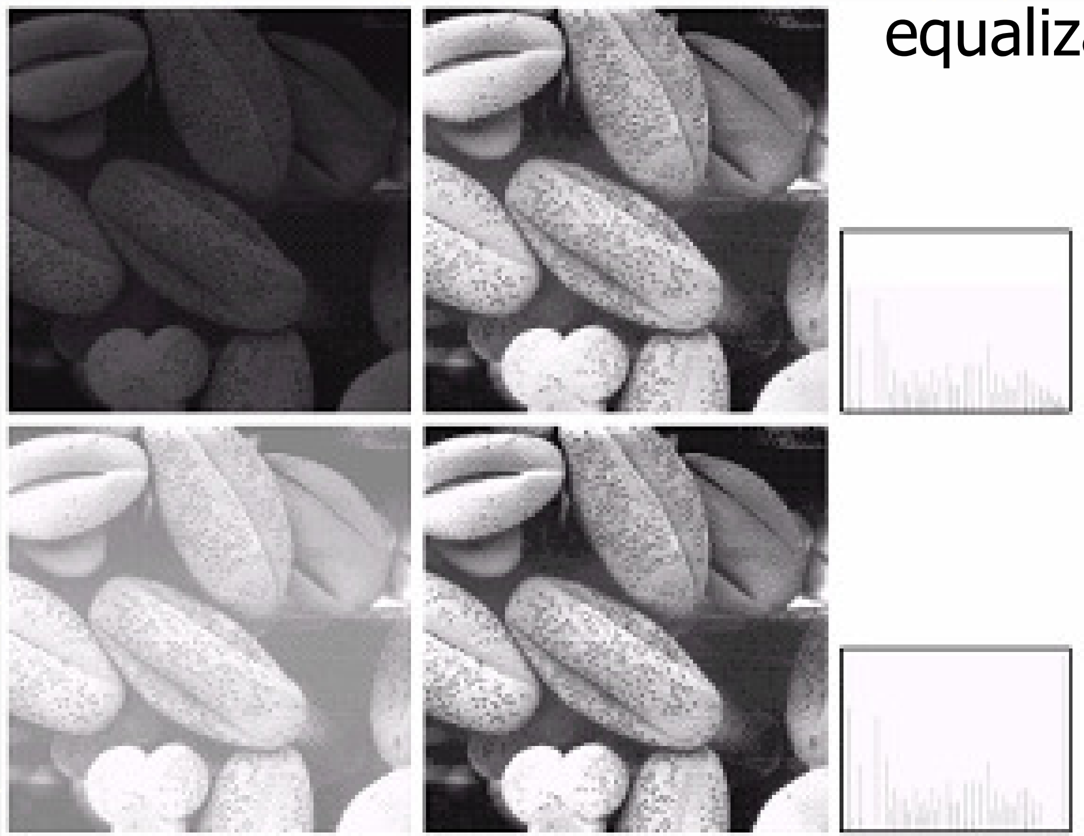
$$= \sum_{j=0}^{k} \frac{n_j}{n} \qquad where\ k = 0, 1, ..., L\text{-}1$$

Histogram Equalization

- Thus, an output image is obtained by mapping each pixel with level r_k in the input image into a corresponding pixel with level s_k in the output image
- In discrete space, it cannot be proved in general that this discrete transformation will produce the discrete equivalent of a uniform probability density function, which would be a uniform histogram

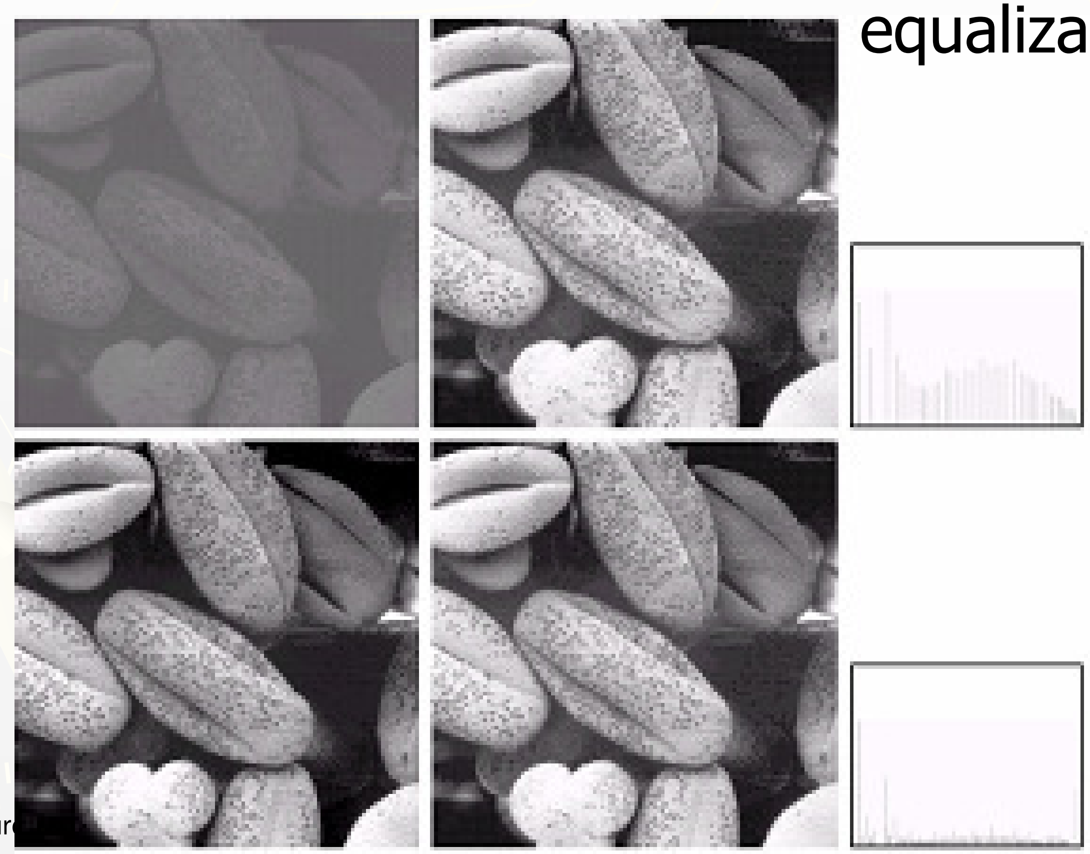
Example

before after Histogram equalization



Example

before after Histogram equalization



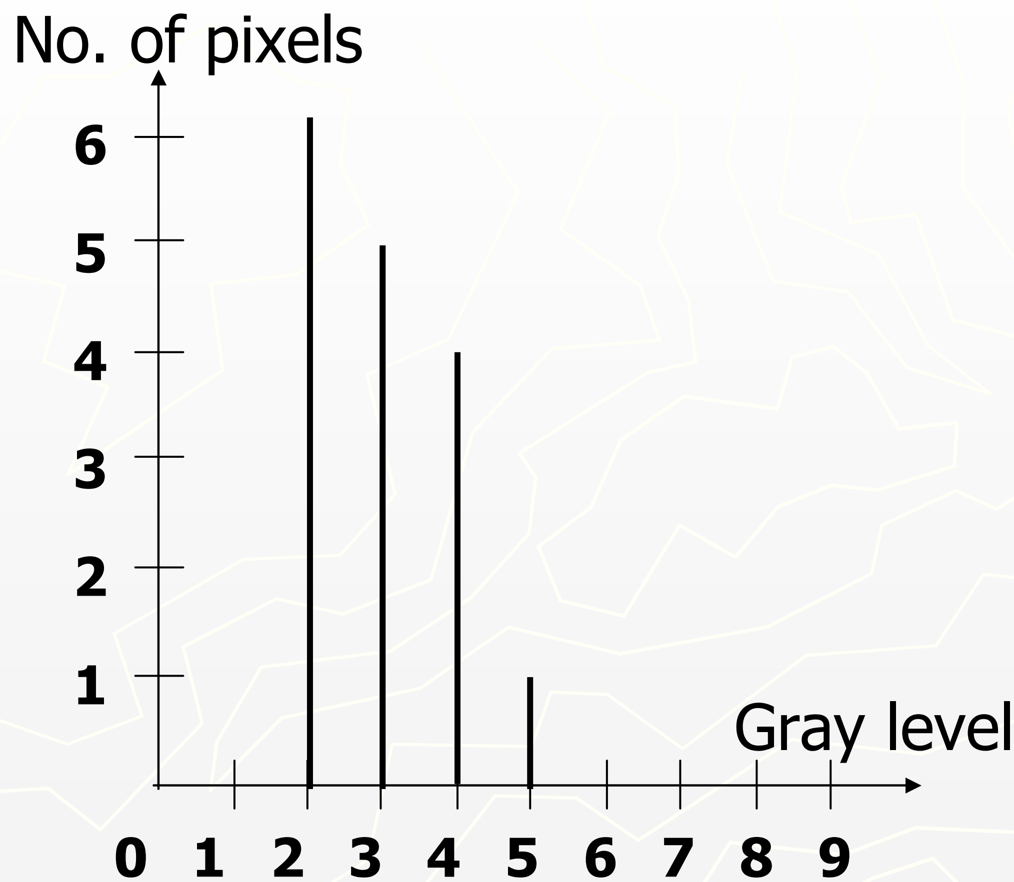
The quality is not improved much because the original image already has a broaden gray-level scale

Example

2	3	3	2
4	2	4	3
3	2	3	5
2	4	2	4

4x4 image

Gray scale = [0,9]



histogram

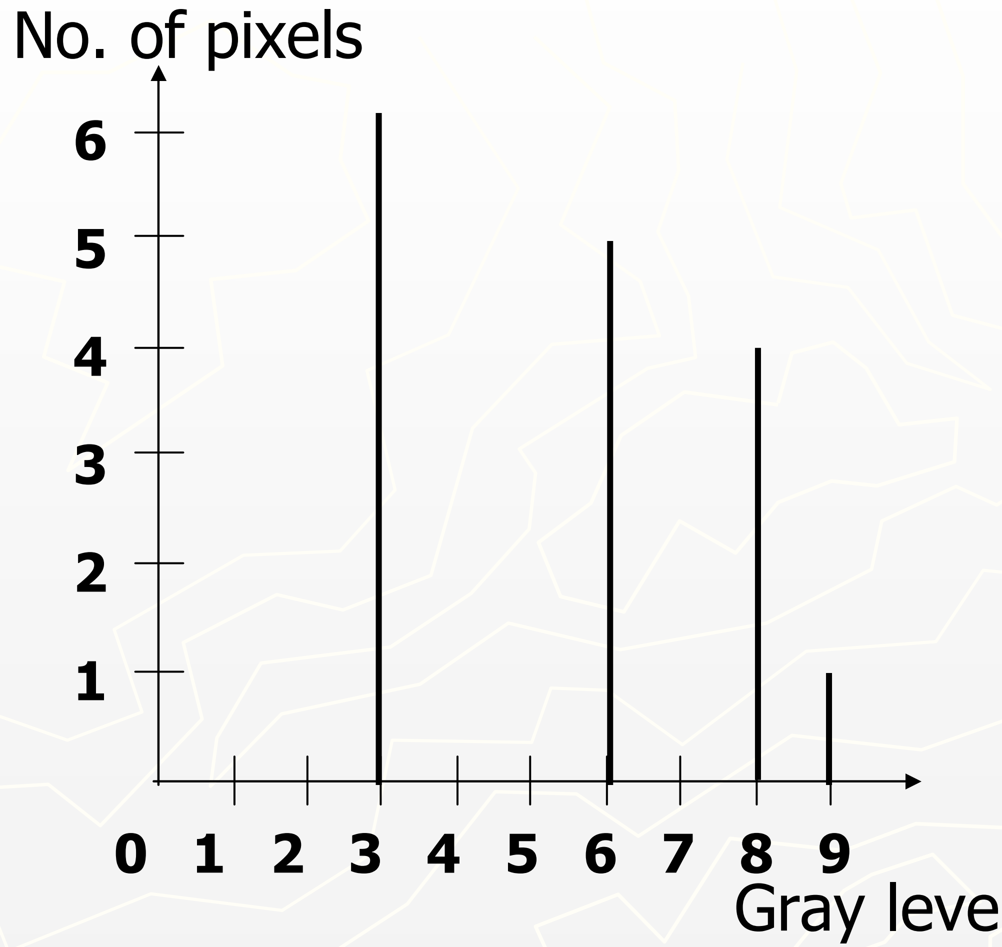
Gray Level(j)	0	1	2	3	4	5	6	7	8	9
No. of pixels	0	0	6	5	4	1	0	0	0	0
$\sum_{j=0}^{k} n_j$	0	0	6	11	15	16	16	16	16	16
$s = \sum_{j=0}^{k} \frac{n_j}{n}$	0	0	6 / 16	11 / 16	15 / 16	16 / 16	16 / 16	16 / 16	16 / 16	16 / 16
s x 9	0	0	3.3 ≈3	6.1 ≈6	8.4 ≈8	9	9	9	9	9

Example

3	6	6	3
8	3	8	6
6	3	6	9
3	8	3	8

Output image

Gray scale = [0,9]



Histogram equalization

Note

- It is clearly seen that
  - Histogram equalization distributes the gray level to reach the maximum gray level (white) because the cumulative distribution function equals 1 when $0 \le r \le L-1$
  - If the cumulative numbers of gray levels are slightly different, they will be mapped to little different or same gray levels as we may have to approximate the processed gray level of the output image to integer number
  - Thus the discrete transformation function can't guarantee the one to one mapping relationship

Histogram Matching (Specification)

- Histogram equalization has a disadvantage which is that it can generate only one type of output image.
- With Histogram Specification, we can specify the shape of the histogram that we wish the output image to have.
- It doesn't have to be a uniform histogram

Consider the continuous domain

Let $p_r(r)$ denote continuous probability density function of gray-level of input image, r

Let $p_z(z)$ denote desired (specified) continuous probability density function of gray-level of output image, z

Let s be a random variable with the property

$$s = T(r) = \int_0^r p_r(w)\,dw$$ ⇒ Histogram equalization

Where w is a dummy variable of integration

Next, we define a random variable z with the property

$$g(z) = \int_0^z p_z(t)\,dt = s$$ ⇒ Histogram equalization

Where t is a dummy variable of integration

thus

$$s = T(r) = G(z)$$

Therefore, z must satisfy the condition

$$z = G^{-1}(s) = G^{-1}[T(r)]$$

Assume G^{-1} exists and satisfies the condition (a) and (b)

We can map an input gray level r to output gray level z

Procedure Conclusion

1. Obtain the transformation function T(r) by calculating the histogram equalization of the input image

$$s = T(r) = \int_0^r p_r(w)\,dw$$

2. Obtain the transformation function G(z) by calculating histogram equalization of the desired density function

$$G(z) = \int_0^z p_z(t)\,dt = s$$

Procedure Conclusion

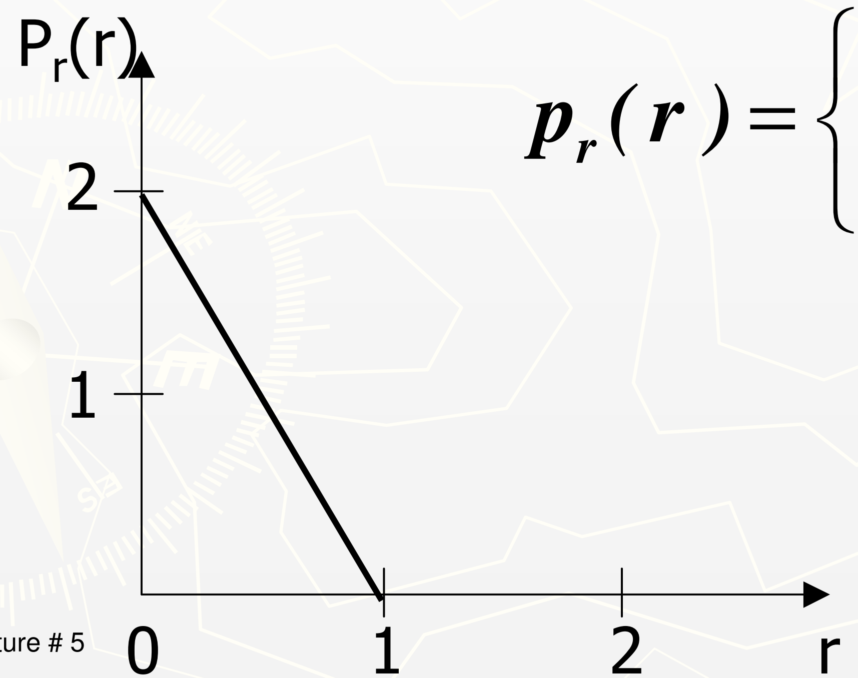
3. Obtain the inversed transformation function G^{-1}

$$z = G^{-1}(s) = G^{-1}[T(r)]$$

4. Obtain the output image by applying the processed gray-level from the inversed transformation function to all the pixels in the input image

Example

Assume an image has a gray level probability density function $p_r(r)$ as shown.

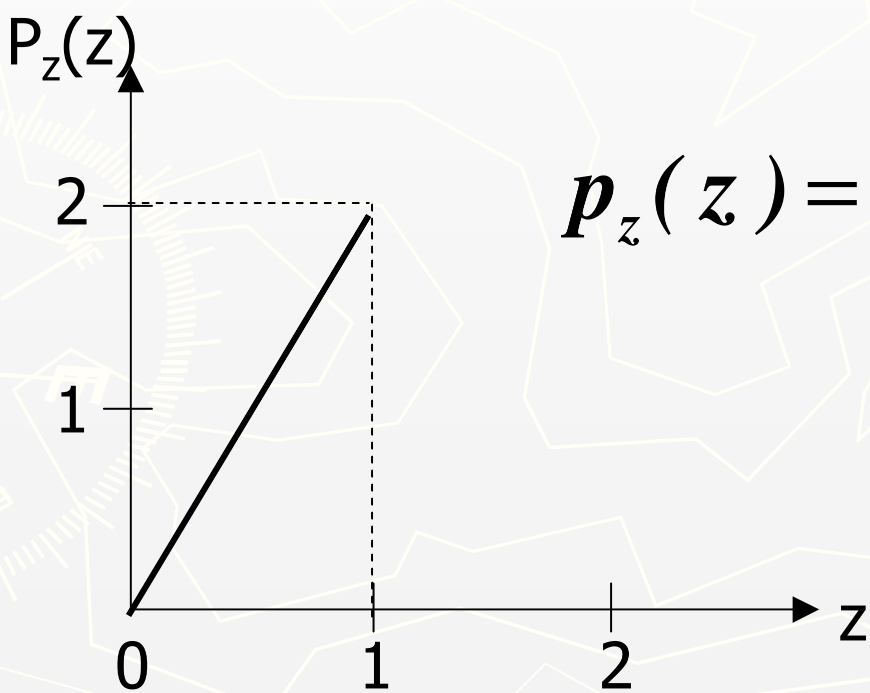


$$p_r(r) = \begin{cases} -2r + 2 & ;0 \le r \le 1 \\ 0 & ;\text{elsewhere} \end{cases}$$

$$\int_0^r p_r(w)dw = 1$$

Example

We would like to apply the histogram specification with the desired probability density function $p_z(z)$ as shown.

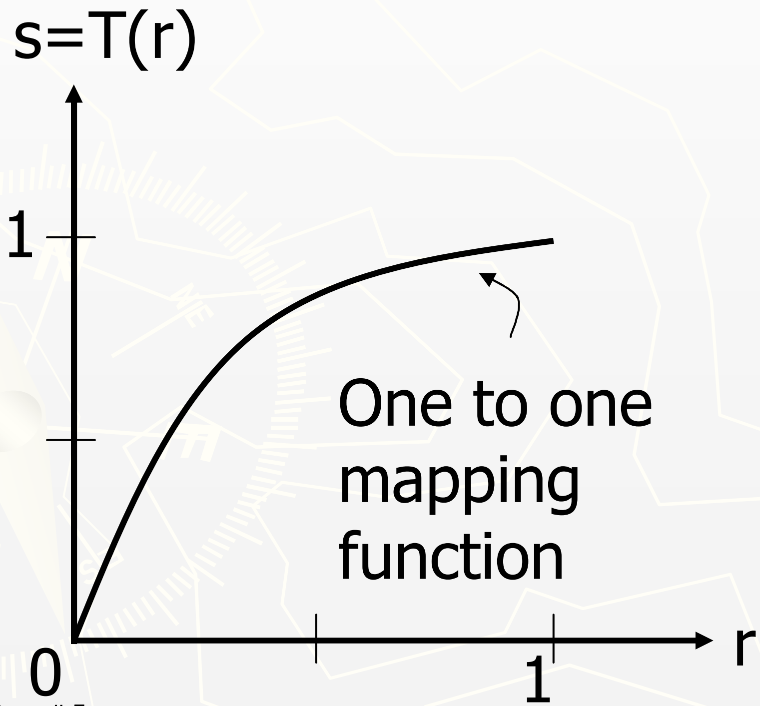


$$p_z(z) = \begin{cases} 2z & ; 0 \le z \le 1 \\ 0 & ; \text{elsewhere} \end{cases}$$

$$\int_0^z p_z(w)\,dw = 1$$

Step 1:

Obtain the transformation function T(r)



$$s = T(r) = \int_0^r p_r(w)dw$$

$$= \int_0^r (-2w + 2)dw$$

$$= -w^2 + 2w\Big|_0^r$$

$$= -r^2 + 2r$$

Step 2:

Obtain the transformation function G(z)

$$G(z) = \int_0^z (2w)dw \quad = z^2\Big|_0^z \quad = z^2$$

Step 3:

Obtain the inversed transformation function G^{-1}

$$G(z) = T(r)$$

$$z^2 = -r^2 + 2r$$

$$z = \sqrt{2r - r^2}$$

We can guarantee that $0 \le z \le 1$ when $0 \le r \le 1$

Discrete formulation

$$s_k = T(r_k) = \sum_{j=0}^{k} p_r(r_j)$$

$$= \sum_{j=0}^{k} \frac{n_j}{n} \qquad k = 0,1,2,\ldots,L-1$$

$$G(z_k) = \sum_{i=0}^{k} p_z(z_i) = s_k \qquad k = 0,1,2,\ldots,L-1$$

$$z_k = G^{-1}\left[T(r_k)\right]$$

$$= G^{-1}\left[s_k\right] \qquad k = 0,1,2,\ldots,L-1$$

Example

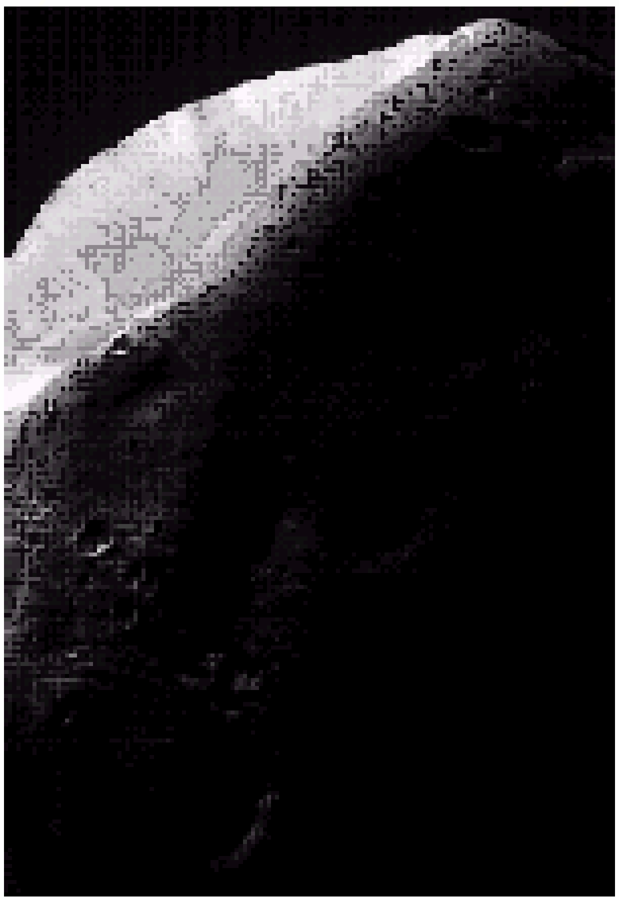


Image of Mars moon

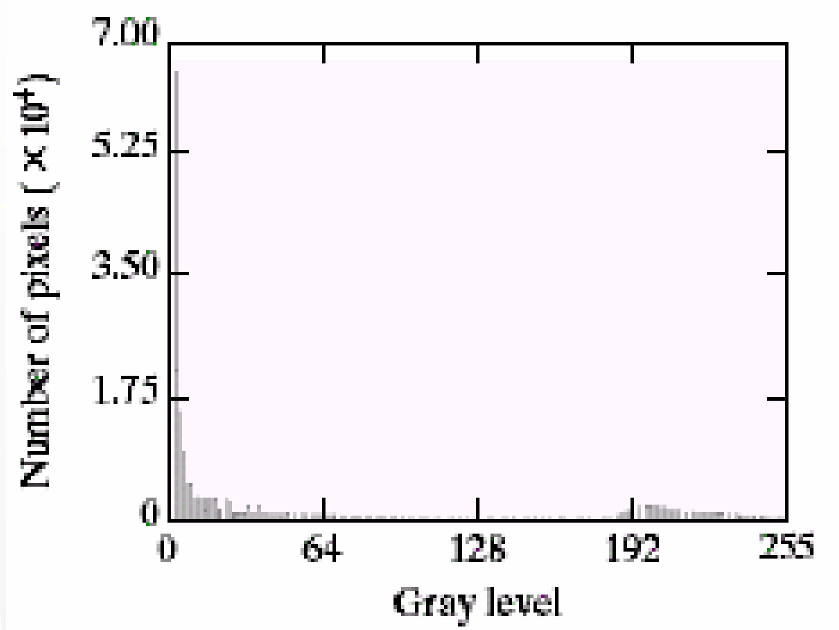
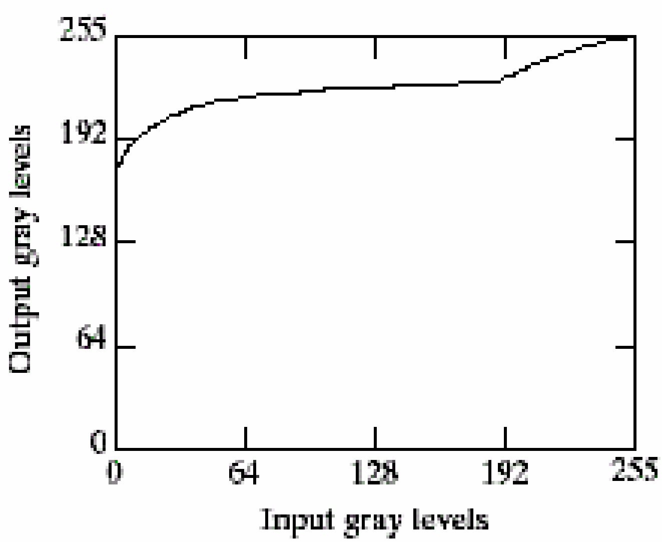
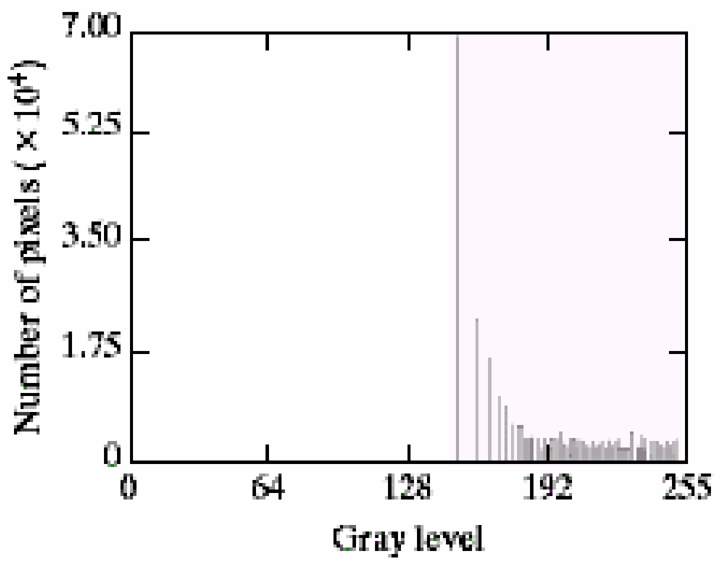


Image is dominated by large, dark areas, resulting in a histogram characterized by a large concentration of pixels in pixels in the dark end of the gray scale

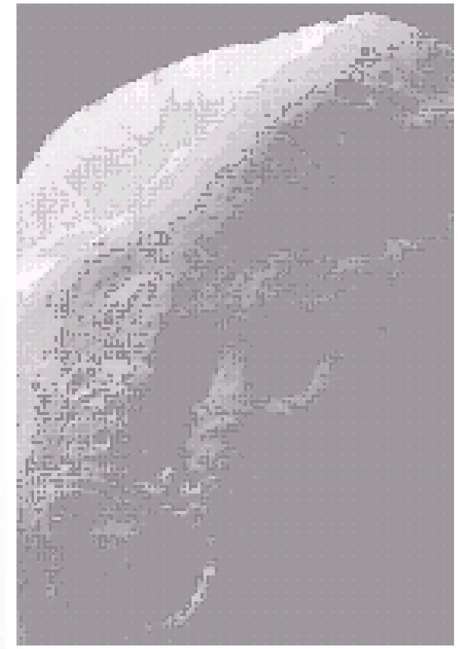
Image Equalization



Transformation function for histogram equalization



Histogram of the result image



Result image after histogram equalization

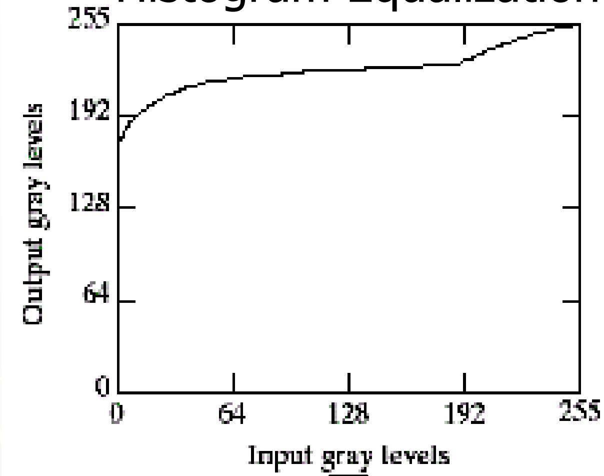
The histogram equalization doesn't make the result image look better than the original image. Consider the histogram of the result image, the net effect of this method is to map a very narrow interval of dark pixels into the upper end of the gray scale of the output image. As a consequence, the output image is light and has a washed-out appearance.

Solve the problem

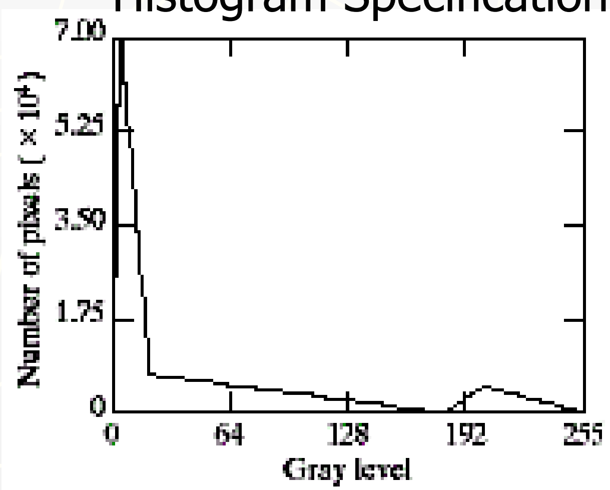
Since the problem with the transformation function of the histogram equalization was caused by a large concentration of pixels in the original image with levels near 0

a reasonable approach is to modify the histogram of that image so that it does not have this property

Histogram Equalization



Histogram Specification



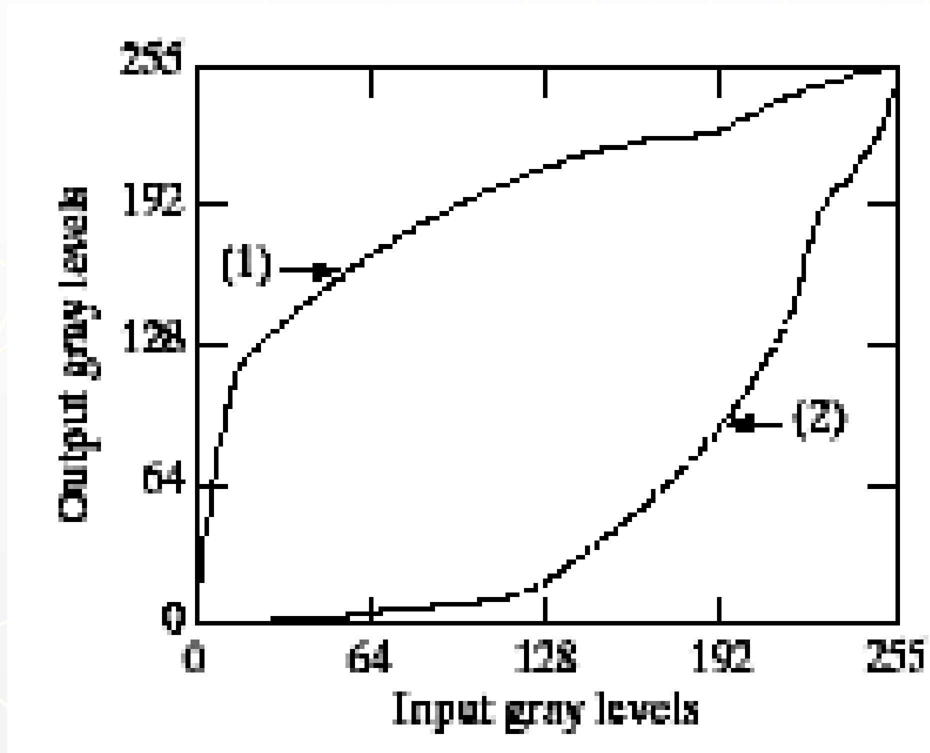
Histogram Specification

- (1) the transformation function G(z) obtained from

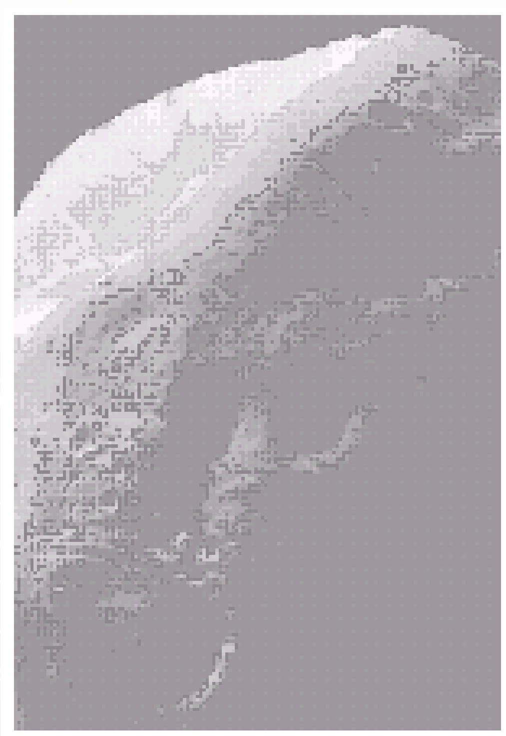
$$G(z_k) = \sum_{i=0}^{k} p_z(z_i) = s_k$$

$$k = 0,1,2,\ldots, L-1$$

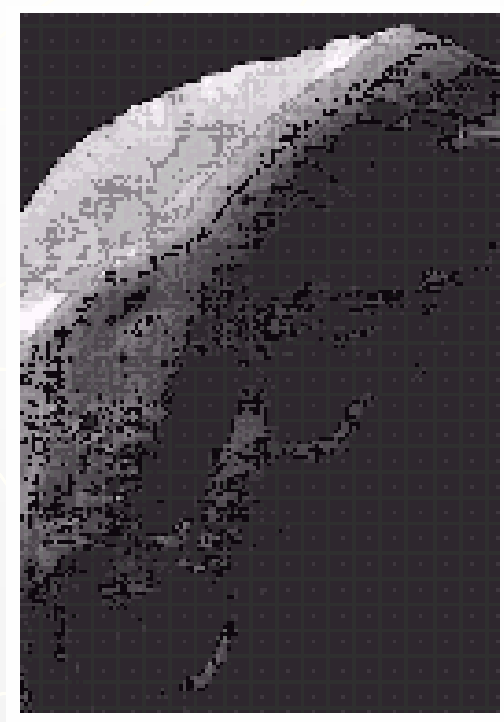
- (2) the inverse transformation $G^{-1}(s)$



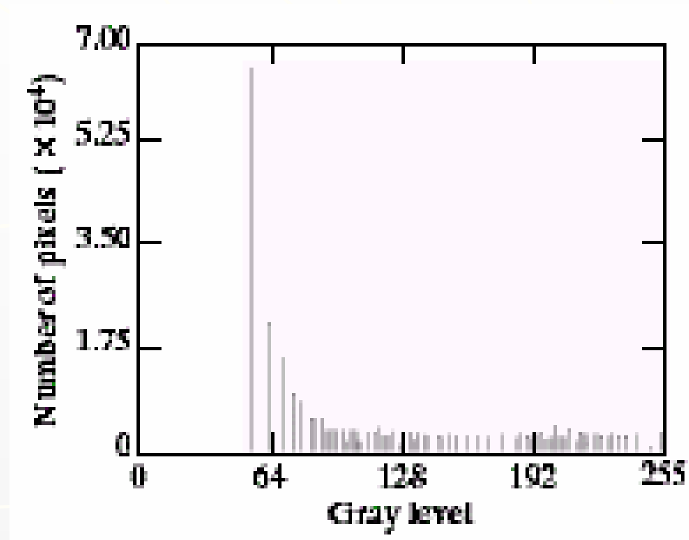
Result image and its histogram



Original image



After applied the histogram equalization



The output image's histogram

Notice that the output histogram's low end has shifted right toward the lighter region of the gray scale as desired.

Note

- Histogram specification is a trial-and-error process
- There are no rules for specifying histograms, and one must resort to analysis on a case-by-case basis for any given enhancement task.

Note

- Histogram processing methods are global processing, in the sense that pixels are modified by a transformation function based on the gray-level content of an entire image.
- Sometimes, we may need to enhance details over small areas in an image, which is called a local enhancement.

Local Histogram Processing

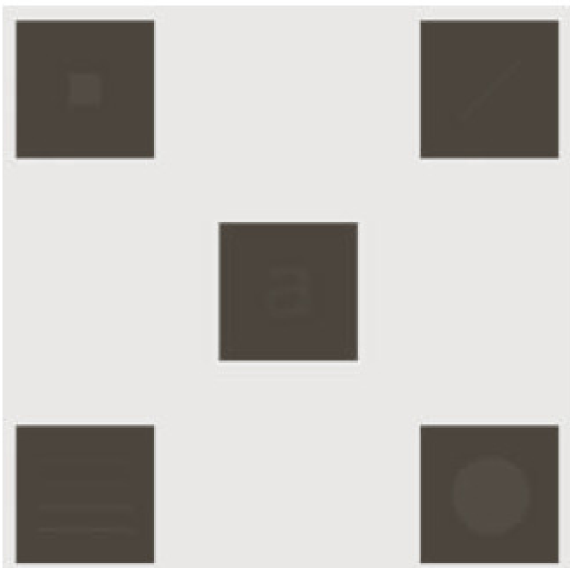
Define a neighborhood and move its center from pixel to pixel

At each location, the histogram of the points in the neighborhood is computed. Either histogram equalization or histogram specification transformation function is obtained

Map the intensity of the pixel centered in the neighborhood

Move to the next location and repeat the procedure

Local Histogram Processing: Example



a b c

FIGURE 3.26 (a) Original image. (b) Result of global histogram equalization. (c) Result of local histogram equalization applied to (a), using a neighborhood of size 3 × 3.